

**A BANACH REARRANGEMENT NORM CHARACTERIZATION
FOR TAIL BEHAVIOR OF MEASURABLE FUNCTIONS
(RANDOM VARIABLES).**

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ABSTRACT.

We construct a Banach rearrangement invariant norm on the measurable space for which the finiteness of this norm for measurable function (random variable) is equivalent to suitable tail (heavy tail and light tail) behavior.

We investigate also a conjugate to offered spaces and obtain some embedding theorems.

Possible applications: Functional Analysis (for instance, interpolation of operators), Integral Equations, Probability Theory and Statistics (tail estimations for random variables) etc.

Key words and phrases: Tail function, rearrangement invariant norm, slowly varying functions, random variable, fundamental function, embedding theorem, conjugate, dual and associate spaces, natural weight and space, distributions, weight, exponential and ordinary Young-Orlicz function, light and heavy tails, upper and lower estimates, left inverse function, Lebesgue spaces, weak and strong Orlicz, Lorentz, Marcinkiewicz norm and spaces.

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1 Notations. Statement of problem.

Let $(X = \{x\}, \mathcal{A}, \mu)$ be measurable space with non-trivial sigma-finite measure μ . We will suppose without loss of generality in the case $\mu(X) < \infty$ that $\mu(X) = 1$ (the probabilistic case) and denote $x = \omega$, $\mathbf{P} = \mu$.

Define as usually for arbitrary measurable function $f : X \rightarrow R$ its distribution function (more exactly, tail function)

$$T_f(t) = \mu\{x : |f(x)| \geq t\}, \quad t \geq 0,$$

$$\|f\|_p = \left[\int_X |f(x)|^p \mu(dx) \right]^{1/p}, \quad p \geq 1; \quad L_p = \{f, \|f\|_p < \infty\},$$

and denote by $f^*(t) = T_f^{-1}(t)$ the left inverse to the tail function $T_f(t)$;

$$f^{**}(t) \stackrel{\text{def}}{=} t^{-1} \int_0^t f^*(s) ds, \quad t > 0.$$

We will denote the set of all tail functions as $\{T\}$; obviously, the set $\{T\}$ contains on all the functions $\{H = H(t), t \geq 0\}$ which are right continuous, monotonically non-increasing with values in the set $[0, \mu(X)]$.

Let $w = w(s), s \geq 0$ be any continuous strictly increasing numerical function (weight) defined on the set $s \in (0, \mu(X))$ such that

$$w(s) = 0 \Leftrightarrow s = 0; \quad \lim_{s \rightarrow \mu(X)} w(s) = \infty. \quad (1.1)$$

In what follows the variables $s, T = T(t)$ changes in the interval $0 < s, T < \mu(X)$.

Moreover, we impose on the set of all such a functions $W = \{w\}$ the following restriction:

$$\forall w \in W \exists T \in \{T\} \Rightarrow w(T(s)) = 1/s. \quad (1.2)$$

Let us introduce the following important functional

$$\gamma(w) = \sup_{t>0} \left[\frac{w(t)}{t} \int_0^t \frac{du}{w(u)} \right] \quad (1.3)$$

and the following quasi-norms:

$$\|f\|_w^* = \sup_{t>0} [w(t) f^*(t)], \quad (1.4)$$

$$\|f\|_w = \sup_{t>0} [w(t) f^{**}(t)], \quad (1.5)$$

The necessary and sufficient condition for finiteness of the functional $\gamma(w)$ see, e.g. in the article [1].

Remark 1.1. Note that

$$\|f\|_w^* = \sup_{t>0} [tw(T_f(t))], \quad (1.6)$$

so that if $\|f\|_w^* \in (0, \infty)$, then

$$T_f(t) \leq w^{-1}(\|f\|_w^*/t).$$

Therefore the functional $f \rightarrow \|f\|_w^*$ may called "the tail quasinorm".

Analogous functionals was introduced in the books [9], chapter 1; [13], chapter 9; in the articles [15], [16], [17] etc. For instance, in the article [15] was introduced and investigated the functionals

$$\rho_{\Phi}(f) = \sup_{t>0} [\Phi(t) T_f(t)], \quad (1.7)$$

$$\|f\|_{wL\Phi} = \inf\{c, c > 0, \sup_{t>0} [\Phi(t/c) T_f(t)] \leq 1\} \quad (1.8)$$

and the spaces $wL\Phi = \{f : \|f\|_{wL\Phi} < \infty\}$,

$$wM\Phi = \{f : \forall c > 0 \Rightarrow \sup_{t>0} [\Phi(t/c) T_f(t)] < \infty\}. \quad (1.9)$$

Here $\Phi(\cdot)$ is arbitrary Young-Orlicz function.

The spaces $wM\Phi, wL\Phi$ was named in [5], [6], [15] as "weak Orlicz spaces," in [9], [13] as "weak Lebesgue spaces". The functionals of a type (1.5) - (1.9) are called "Lorentz weak norm" or "Marcinkiewicz norm", see [1], [4], [11], [18], [22] etc.

In the article [4] was considered more general case of a functionals of a view (in our notations)

$$\|f\|_{w,p} = \left[\int_0^\infty f^*(t) w(t) dt \right]^{1/p},$$

$\|f\|_{w,\infty} = \sup_t [f^*(t)w(t)]$ and was obtained in particular the condition for quasi-normalizing of this spaces.

For instance, if $\Phi(t) = \Phi_p(t) = t^{1/p}, p \in [1, \infty)$, then both the spaces $wM\Phi, wL\Phi$ coincides with the Lorentz space $L(p, \infty)$, see [2], chapter 4, section 4, p. 216-217.

Remark 1.2. As long as

$$f^{**}(t) = t^{-1} \sup_{\mu(E) \leq t} \int_E |f(x)| \mu(dx),$$

we can rewrite the expression for $\|f\|_w$ as follows:

$$\|f\|_w = \sup_{t>0} \left[(w(t)/t) \cdot \sup_{E: \mu(E) \leq t} \int_E |f(x)| \mu(dx) \right]. \quad (1.10)$$

If the measure μ has not atoms, then the expression (1.10) may be rewritten as follows:

$$\|f\|_w = \sup_{E: 0 < \mu(E) < \infty} \left[\frac{w(\mu(E))}{\mu(E)} \cdot \int_E |f(x)| \mu(dx) \right]. \quad (1.10a)$$

It follows from equality (1.10) that $\|f\|_w$ is true rearrangement invariant norm and the space $L_w = \{f : \|f\|_w < \infty\}$ is complete Banach functional rearrangement invariant space with Fatou property. The proof is alike to one in the case $w(t) = t^{1/p}, p \geq 1$; see [2], chapters 1,2; [23], chapter 8.

The norm $\|f\|_w$ is named Marcinkiewicz's norm, see [11], chapter 2, section 2.

Example 1.1. Fundamental function.

Let δ be arbitrary number from the set $(0, \mu(X))$ and let B be any measurable set such that $\mu(B) = \delta$. The function $\phi(\delta; Z)$ defined aside from δ on the rearrangement invariant space Z as follows:

$$\phi(\delta; Z) = \|I(B)\|Z$$

is called a fundamental function of the space Z .

If the measure μ has not atoms, then the formula (1.10a) gives us:

$$\phi(\delta; L_w) = w(\delta).$$

Remark 1.3. The equality (1.6) may be used for the following definition. Let g be absolutely integrable non-zero function: $g \in L_1(X, \mu)$. We define the *natural* weight function $w^{(g)} = w^{(g)}(s)$ as follows:

$$1 = t \cdot w^{(g)}(T_g(t)), \quad (1.11)$$

or equally

$$w^{(g)}(s) = T_g^{-1}(1/s). \quad (1.12)$$

Note that $\|g\|_{w^{(g)}}^* = 1$.

We will say that in this case the function $g(\cdot)$ generated the correspondent space $L_{w^{(g)}}$ and $L_w^{(g,*)} = \{f : \|f\|_{w^{(g)}}^* < \infty\}$.

Our aim in this short report is to prove (under simple conditions) that the quasinorm $\|f\|_w^*$ and the norm $\|f\|_w$ are linear equivalent.

This fact is true for heavy tails; the case of light tails will be considered further, in which we prove that the weak Orlicz's norm is equivalent ordinary Orlicz's norm for exponential correspondent Young function.

Our results improve ones in [1], [11], chapter 2, section 2; see also reference therein. In particular, we find the exact value for estimated functional (Theorem 2.1), consider the case of light tails (Theorem 3.1), generalize the embedding theorem in [12], p. 167 etc.

2 Main result: the case of heavy tails.

Theorem 2.1. Let

$$w \in W, \quad \gamma(w) < \infty, \quad (2.1)$$

then

$$1 \cdot \|f\|_w^* \leq \|f\|_w \leq \gamma(w) \cdot \|f\|_w^*, \quad (2.2)$$

and both the coefficients "1" and " $\gamma(w)$ " in (2.2) are the best possible.

Proof is at the same as for the spaces $L(p, \infty)$, see [23], chapter 8.

A. Inequalities.

The left-hand side of assertion (2.2) follows immediately from the inequality $f^{**}(t) \geq f^*(t)$ even without the conditions (2.1).

Let now $\gamma(w) < \infty$. We have:

$$\begin{aligned}
w(t)f^{**}(t) &= w(t) \frac{1}{t} \int_0^t f^*(u)du = \frac{w(t)}{t} \int_0^t \frac{w(u)f(u)du}{w(u)} \leq \\
\frac{w(t)}{t} \|f\|_w^* \int_0^t \frac{du}{w(u)} &\leq \gamma(w) \|f\|_w^*.
\end{aligned} \tag{2.3}$$

Taking supremum of the last inequality over all the values t , we obtain the right-hand side (2.2).

B. Exactness.

The exactness of the constant "1" follows immediately from the consideration of the case $w = w_p$, $p > 1$. Namely, in this case we obtain the classical inequality

$$\|f\|_{w_p}^* \leq \|f\|_{w_p} \leq \frac{p}{p-1} \cdot \|f\|_{w_p}^*;$$

note that $\lim_{p \rightarrow \infty} p/(p-1) = 1$.

It remains to prove the exactness of the coefficient $\gamma(w)$ in the right-hand side of inequality (2.2). It is reasonable to suppose $\gamma(w) < \infty$; in other case it is nothing to prove.

Let here $w \in W$; we can prove moreover that the right-hand inequality in (2.2) is exact still for arbitrary function w .

There exists a positive (measurable) integrable function $g : X \rightarrow R_+$ for which

$$w(t) = w^{(g)}(t) = 1/T_g^{-1}(t) = 1/g^*(t), \tag{2.4}$$

then $\|g\|_{w^{(g)}}^* = 1$ and both the spaces L_w , L_w^* are generated by means of the function g .

Denote for any function $w \in W$ the following functional

$$G(w) := \sup_{f: \|f\|_w^* = 1} \left[\frac{\|f\|_w}{\|f\|_w^*} \right] = \sup_{f: \|f\|_w = 1} \|f\|_w. \tag{2.5}$$

We have:

$$\begin{aligned}
G(w) &\geq \|g\|_w = \sup_t [w(t)g^{**}(t)] = \sup_t \left[w(t) \cdot \frac{1}{t} \int_0^t g^*(s)ds \right] = \\
&\sup_t \left[w(t) \cdot \frac{1}{t} \int_0^t \frac{ds}{w(s)} \right] = \gamma(w),
\end{aligned}$$

Q.E.D.

Example 2.1. We suppose in all considered in this article examples that $\mu = \mathbf{P}$ or equally $\mu(X) = 1$.

Let $w(s) = w_{p,l}(s) = s^{1/p} l(s)$, $1 \leq p < \infty$, where $l = l(s)$ is non-negative positive for positive values s continuous on the semi-open interval $0 < s \leq 1$, slowly varying function as $s \rightarrow 0+$. By definition, $w_p(s) = w_{p,1}(s) = s^{1/p}$.

The condition (2.1) is satisfied iff $p > 1$.

The asymptotical as $t \rightarrow 0+$ relation

$$I := \int_0^t \frac{du}{w_{p,l}(u)} \asymp C \frac{t}{w_{p,l}(t)}, \quad p > 1 \quad (2.6)$$

may be obtained from the book [3], chapter 1, sections 1.5., 1.6 p. 26-27.

As a consequence: for a random variable $\xi = \xi(\omega)$ the tail inequality

$$T_\xi(t) \leq C_1 K^p t^{-p} \log^{-\kappa p}(t/K), \quad t \geq 2K,$$

$$C_1, K = \text{const} > 0, \quad \kappa = \text{const}, p = \text{const} > 1 \quad (2.7)$$

is equivalent to the following norm estimation:

$$\sup_{0 < t \leq 1/2} \left\{ t^{1/p-1} |\log^\kappa(t)| \sup_{0 < \mathbf{P}(E) \leq t} \int_E |\xi(\omega)| \mathbf{P}(d\omega) \right\} \leq C_2(p, \kappa) K. \quad (2.8)$$

If the measure \mathbf{P} is atomless, then the inequality (2.8) may be simplified as follows:

$$\sup_{0 < \mathbf{P}(E) \leq 1/2} \left\{ [\mathbf{P}(E)]^{1/p-1} |\log^\kappa \mathbf{P}(E)| \int_E |\xi(\omega)| \mathbf{P}(d\omega) \right\} \leq C_2(p, \kappa) K. \quad (2.9)$$

Notice that this norm description of heavy tail distributed random variables is more convenient as description by means of the so-called moment, or Grand Lebesgue spaces, as long as ones are not completely adequate, see the example 5.1 in the article [19].

3 Main result: the case of light tails.

Example 3.1.

Let $X = (0, 1)$ with Lebesgue measure, and let

$$h(x) = |\log x|, \quad x \in X, \quad (3.1)$$

then $T_h(t) = e^{-t}$, $t \geq 0$;

$$h^*(s) = |\log s|; \quad w^{(h)}(s) = |\log s|^{-1}, \quad s \in (0, 1); \quad \|h\|_{w^{(h)}}^* = 1; \quad (3.2)$$

but

$$f^{**}(t) = \frac{1}{t} \cdot \int_0^t |\log x| dx = 1 + |\log t|,$$

and

$$\|h\|_{w^{(h)}} = \sup_{t > 0} [w^{(h)}(t) h^{**}(t)] = \sup_{t \in (0, 1)} \frac{1 + |\log t|}{|\log t|} = \infty. \quad (3.3)$$

Analogous implication $\|h\|_{w^{(h)}}^* = 1$, $\|h\|_{w^{(h)}} = \infty$ if true for the functions of a view

$$h(x) = h_{m,l(\cdot)}(x) = |\log x|^m l(|\log x|), \quad m = \text{const} > 0,$$

$l(z)$ is slowly varying as $z \rightarrow \infty$ non-negative function.

Notice that the functions of a view $h_{m,l(\cdot)}(x)$, $x \in X$ have a light tails. Indeed, they belong to the so-called *exponential* Orlicz spaces.

Recall that the Young-Orlicz function $N = N(u)$, $u \in R$ is called *exponential Young-Orlicz function*, briefly: EOF, if

$$N(u) = e^{\nu(u)} - 1, \quad (3.4)$$

where $\nu = \nu(u)$ is even twice continuous differentiable convex function, strictly monotonically increasing on the right-hand semi-axis and such that

$$\nu(u) = 0 \Leftrightarrow u = 0; \nu'(0) = 0; \lim_{u \rightarrow \infty} \nu'(u) = \infty. \quad (3.5)$$

The Orlicz space $L(N)$ with Young-Orlicz function $N = N(u)$ defined over Probabilistic space $(X, \mathcal{A}, \mathbf{P})$ is said to be *exponential Orlicz space*, briefly: EOS, if the function $N = N(u)$ is exponential Orlicz function EOF.

We denote in accordance with [12], p. 167 as FN the set of all (measurable) functions $u = u(x)$, $u : X \rightarrow R$ such that

$$\forall k > 0 \Rightarrow \int_X N(k|u(x)|) \mu(dx) < \infty. \quad (3.6)$$

Evidently, $F(N) \subset L(N)$.

Further, the subspace $E(N)$ of the whole space $L(N)$ is by definition the closure of the set of all bounded measurable functions supported on the set of finite measure. It is known (see, e.g. [12], p. 167) that $E(N) = F(N)$.

Theorem 3.1. Let the function $\Phi = \Phi(u)$ be EOF, so that the Orlicz space $L(\Phi)$ is EOS. Then:

A. The weak Orlicz space $wL\Phi$ coincides as the set equality with norm equivalence with the ordinary (exponential) Orlicz space $L(\Phi)$.

B. The weak Orlicz space $wM\Phi$ coincides as the set equality with norm equivalence with the subspace $E(\Phi)$.

Proof A. Let $f \in L(\Phi)$, $f \neq 0$; we can suppose without loss of generality $\|f\|_{L(\Phi)} = 1$. Then

$$\int_X \Phi(|f(x)|) \mathbf{P}(dx) \leq 1.$$

We use the Tchebychev's inequality:

$$\mathbf{P}(|f| > C) \leq \frac{1}{\Phi(C)}, \quad C = \text{const} > 0; \quad (3.7)$$

therefore $f \in wL\Phi$ and $\|f\|_{wL\Phi} \leq 1$.

Conversely, let $f \in wL\Phi$ and $\|f\|_{wL\Phi} \leq 1$. It follows from the definition of the functional $f \rightarrow \|f\|_{wL\Phi}$ that

$$T_f(t) \leq \frac{1}{\Phi(t)}, \quad t > 0$$

or equally

$$T_f(t) \leq C_1 e^{-\nu(t)}, \quad t \geq 1. \quad (3.8)$$

It follows from the estimate (3.8) that

$$\exists C_2 = C_2(C_1; \nu(\cdot)) \in (0, \infty) \Rightarrow \int_X \Phi(|f(x)|/C_2) \mathbf{P}(dx) < \infty; \quad (3.9)$$

$$\|f\|L(\Phi) \leq C_3(C_1, C_2) < \infty, \quad (3.10)$$

see [10]; more detail explanation see in a monograph [20], chapter 1, section 1.2.

Proof B. Let $f \in wM\Phi$; we conclude using at the same arguments as before

$$\forall C_4 > 0 \Rightarrow \int_X \Phi(|f(x)|/C_4) \mathbf{P}(dx) < \infty. \quad (3.11)$$

It follows from the proposition (3.11) that the function f belongs to the subspace $F(\Phi)$; but we know $F(\Phi) = E(\Phi)$.

The converse inclusion $E(\Phi) \subset wM\Phi$ follows from the simple verified fact that for every measurable set B with finite measure $\mu(B) < \infty$ its indicator $I(B)$ belongs to the set $wM\Phi$ since

$$T_{I(B)}(t) = 0, \quad t > 1.$$

Recall that the Absolutely Continuous Norm ($ACN = ACN(Z)$) part of the rearrangement invariant (r.i.) space $(Z, \|\cdot\|_Z)$ over the triple $(X, \mathcal{A}, \mathbf{P})$ consists on the functions $\{f\}$ with Absolutely Continuous Norm (ACN):

$$\lim_{\epsilon \rightarrow 0+} \sup_{B \in \mathcal{A}, \mathbf{P}(B) < \epsilon} \|f \cdot I(B)\|_Z = 0. \quad (3.12)$$

The $I(B)$ denotes the indicator function of the (measurable) set B . The ACN part is also closed subspace of the space $(Z, \|\cdot\|_Z)$.

More information about the spaces $E(N)$ and $ACN(Z)$ see in the classical monograph belonging to C.Bennett and R.Sharpley [2], chapter 1, sections 2,3. In particular, it is proved that (in our notations) that $ACN(L(\Phi)) \subset E\Phi$. As a consequence:

Remark 3.1. Every function from the set $wM\Phi$ has absolutely continuous norm in the whole space $L(\Phi)$.

As a contradiction:

Remark 3.2. Both the spaces: $wL\Phi$, and L_w under conditions (2.1) have not ACN property.

Indeed, the space $wL\Phi$ coincides with the Orlicz space $L(\Phi)$ with the Young-Orlicz function not satisfying the Δ_2 condition.

It remains to consider the space $L_{w_p}(0, 1)$, $p > 1$ with Lebesgue measure m . Choose a function $f(x) = x^{-1/p}$ and let $b \downarrow 0$, $b < 1$; then $m\{(0, b)\} = b \downarrow 0$;

$$\begin{aligned} \|f \cdot I(0, b)\|_w^* &= \sup_{t \in (0, 1)} \left[\frac{1}{t^{1-1/p}} \int_0^{\min(b, t)} x^{-1/p} dx \right] = \\ &= \frac{p}{p-1} \sup_{t \in (0, 1)} \left[\frac{(\min(b, t))^{1-1/p}}{t^{1-1/p}} \right] = \frac{p}{p-1} \end{aligned}$$

and

$$\lim_{b \downarrow 0} \|f \cdot I(0, b)\|_w^* = \frac{p}{p-1} > 0. \quad (3.13)$$

Remark 3.3. The complete description of the conjugate (= associate or dual) space to the considered here weak spaces: weak Orlicz space $L(\Phi)$ etc. see in the book [24], chapter 11. The dual spaces to the Marcinkiewicz and Lorentz spaces, conditions of its reflexivity and separability, description of compact subsets, for instance, weak $L(p)$, $p \geq 1$ spaces are described, e.g. in the book [11], chapter 2, section 3; in the articles [6], [7] etc.

Remark 3.4.

Let $w \in W$ and let the measure μ be resonant; then the fundamental function of the associate space $(L_w)'$ has a view

$$\phi(\delta; (L_w)') = \frac{\delta}{w(\delta)}. \quad (3.14)$$

Remark 3.5. Obviously, the assertion of theorem 3.1 remains true if instead the weak Orlicz norm stated the functional (1.7) $\rho_\Phi(f) = \sup_{t>0} [\Phi(t) T_f(t)]$.

4 Embedding theorem.

In the book [12], p. 105, theorem 2.18 is proved the following embedding theorem. Let $p > 1$, $\epsilon = \text{const} \in (0, p-1)$; then

$$L_p \subset L_{w,p} \subset L_{p-\epsilon}. \quad (4.1)$$

Hereafter in this section the symbol \subset denotes the linear continuous embedding.

We intend here to generalize the assertion (4.1) on the case of general L_w spaces over probabilistic measure space. We consider the case when for $p_0 = \text{const} > 1$, $\Delta = \text{const} \geq 0$,

$$\Phi(u) \stackrel{\text{def}}{=} \Phi_{p_0, \Delta, S}(u) = |u|^{p_0} (\log |u|)^{-\Delta} S(\log |u|), \quad |u| \geq e, \quad (4.2)$$

$$\Phi_{p_0, \Delta, S}(u) = C(p_0, \Delta) u^2, \quad |u| \leq e; \quad C(p_0, \Delta) e^2 = e^{p_0} S(1),$$

where $S = S(z)$, $z \geq 1$ is continuous positive slowly varying as $z \rightarrow \infty$ function.

Note that the function $\Phi_{p_0, \Delta}(u)$ satisfies the Δ_2 condition; the case of exponential Orlicz space $L(\Phi)$ is here trivial.

In order to formulate our result, we need to recall some facts about the so-called Grand Lebesgue spaces (GLS). Recently, see [8], [10], [14], [19] etc. appears the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B)$, $A, B = \text{const}$, $A \geq 1$, $A < B \leq \infty$, spaces consisting on all the measurable functions $f : T \rightarrow R$ with finite norms

$$\|f\|G(\psi) \stackrel{\text{def}}{=} \sup_{p \in (A, B)} \|f\|_p / \psi(p). \quad (4.3)$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (A, B) function such that

$$\inf_{p \in (A, B)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (A, B).$$

We will denote

$$\text{supp}(\psi) \stackrel{\text{def}}{=} (A, B) = \{p : \psi(p) < \infty, \} \quad (4.4)$$

The set of all ψ functions with support $\text{supp}(\psi) = (A, B)$ will be denoted by $\Psi(A, B)$.

These spaces are rearrangement invariant and are used, for example, in the theory of probability, theory of Partial Differential Equations, functional analysis, theory of Fourier series, theory of martingales, mathematical statistics, theory of approximation etc.

We will use the following important example (more exactly, the *families of examples*) of the ψ functions and correspondingly the GLS spaces.

Let us denote

$$\psi(B, \beta; p) \stackrel{\text{def}}{=} (B - p)^{-\beta}, \quad (4.5)$$

where $\beta = \text{const} \geq 0, 1 < B < \infty; p \in [1, B)$ so that

$$\text{supp} \psi(B, \beta; \cdot) = [1, B).$$

As a particular case: let us introduce the following ψ function:

$$\psi_{p_0, \Delta, S}(p) = (p_0 - p)^{-(1+\Delta)/p_0} S^{1/p_0} \left(\frac{p_0}{p_0 - p} \right), \quad (4.6)$$

$\Delta = \text{const} > -1, p_0 = \text{const} > 1; 1 \leq p < p_0$ and the following spaces: $L(\Phi_{p_0, \Delta, S}), wL\Phi_{p_0, \Delta, S}$ and $G\psi_{p_0, \Delta, S}$.

Remark 4.1. If we define the *degenerate* $\psi_r(p), r = \text{const} \geq 1$ function as follows:

$$\psi_r(p) = \infty, \quad p \neq r; \psi_r(r) = 1$$

and agree $C/\infty = 0, C = \text{const} > 0$, then the $G\psi_r(\cdot)$ space coincides with the classical Lebesgue space L_r .

Remark 4.2. Let $\xi : X \rightarrow R$ be some (measurable) function from the set $L(p_1, p_2) = \cup_{p \in (p_1, p_2)} L(p), 1 \leq p_1 < p_2 \leq \infty$. We can introduce the so-called *natural* choice $\psi_\xi(p)$ as follows:

$$\psi_\xi(p) \stackrel{\text{def}}{=} \|\xi\|_p; \quad p \in (p_1, p_2).$$

Evidently, in the case when $\mu(X) < \infty$, then by virtue of Lyapunov's inequality $L(p_1, p_2) = L[1, p_2)$.

Analogously, let $\xi = \{\xi(y)\}, y \in Y$ be some *family* of measurable function uniformly from the set $L(p_1, p_2) = \cup_{p \in (p_1, p_2)} L(p), 1 \leq p_1 < p_2 \leq \infty$. We can introduce the so-called *natural* choice $\psi_{\{\xi\}}(p)$ as follows:

$$\psi_{\{\xi\}}(p) \stackrel{\text{def}}{=} \sup_{y \in Y} \|\xi(y)\|_p; \quad p \in (p_1, p_2).$$

Theorem 4.1.

$$L(\Phi_{p_0, \Delta, S}) \subset wL\Phi_{p_0, \Delta, S} \subset G\psi_{p_0, \Delta, S}. \quad (4.7)$$

Proof. 1. Left hand-side inclusion.

Let $f \in L(\Phi_{p_0, \Delta, S})$, $f \neq 0$; then for some finite positive constant $u \in (0, \infty)$

$$\int_X \Phi_{p_0, \Delta, S}(|f(x)|/u) \mathbf{P}(dx) \leq 1.$$

We use the Tchebychev's inequality:

$$\mathbf{P}(|f(x)|/u > C) \leq \frac{1}{\Phi_{p_0, \Delta, S}(C/u)}, \quad C \in (0, \infty),$$

or equally

$$\mathbf{P}(|f(x)| > C_1) \leq \frac{1}{\Phi_{p_0, \Delta, S}(C_1)}, \quad C_1 \in (0, \infty), \quad (4.8)$$

The last inequality (4.8) imply that $f \in wL\Phi_{p_0, \Delta, S}$.

2. Right hand-side inclusion.

Let $f \in wL\Phi_{p_0, \Delta, S}$. We can and will suppose without loss of generality that

$$T_f(t) \leq t^{-p_0} \log^\Delta(t) S(\log t), \quad t > 2. \quad (4.9)$$

Let $1 \leq p < p_0$. Recall that in this section $\mathbf{P}(X) = 1$, therefore $T_f(t) \leq 1$. As long as

$$\int_X |f(x)|^p \mathbf{P}(dx) = p \int_0^\infty t^{p-1} T_f(t) dt, \quad (4.10)$$

we obtain substituting the estimate (4.9) into (5.10), denoting $\epsilon = p_0 - p$ and tacking into account that $\epsilon \rightarrow 0+$:

$$\begin{aligned} \int_X |f(x)|^p \mathbf{P}(dx) &\leq p_0 \int_0^2 2^{p_0-1} dt + \int_2^\infty t^{p-1} T_f(t) dt; \\ \int_X |f(x)|^p \mathbf{P}(dx) - C_2 &\leq \int_1^\infty t^{p-p_0-1} \log^\Delta(t) S(\log t) dt = \\ &\int_0^\infty e^{-\epsilon y} y^\Delta S(y) dy = \epsilon^{-1-\Delta} \int_0^\infty e^{-x} x^\Delta S(x/\epsilon) dx \sim \\ &\epsilon^{-1-\Delta} \int_0^\infty e^{-x} x^\Delta S(1/\epsilon) dx = \epsilon^{-1-\Delta} S(1/\epsilon) \Gamma(1 + \Delta). \end{aligned}$$

Thus, we have as $p \rightarrow p_0 - 0$

$$\|f\|_p \leq C (p_0 - p)^{-(1+\Delta)/p_0} S^{1/p_0}(1/(p_0 - p)) \Gamma(1 + \Delta) \asymp \psi_{p_0, \Delta, S}(p). \quad (4.11)$$

The detail grounding of passage to limit see in [14], [19].

This completes the proof of theorem 4.1.

In order to show the exactness of proposition the theorem 4.1, let us consider the following inverse result.

Theorem 4.2. We assert under at the same conditions as in theorem 4.1 that

$$G\psi_{p_0,\Delta,S} \subset wL\Phi_{p_0,\Delta+1,S}. \quad (4.12)$$

Proof. Let $f \in G\psi_{p_0,\Delta,S}$; this imply by definition of the norm in Grand Lebesgue Spaces for the values $1 \leq p < p_0$

$$\|f\|_p \leq C \psi_{p_0,\Delta,S}(p) = C(p_0 - p)^{-(1+\Delta)/p_0} S^{1/p_0} \left(\frac{p_0}{p_0 - p} \right). \quad (4.13)$$

We can and will suppose in (4.13) without loss of generality $C = 1$.

By means of Tchebychev's inequality

$$T_f(t) \leq \frac{\psi_{p_0,\Delta,S}^p(p)}{t^p}, \quad t > 0, \quad (4.14)$$

therefore

$$T_f(t) \leq \inf_{1 \leq p < p_0} \left[\frac{\psi_{p_0,\Delta,S}^p(p)}{t^p} \right] \leq C_2 t^{-p_0} \log^{\Delta+1}(t) S(\log t), \quad t > 2. \quad (4.15)$$

The last inequality implies that $f \in wL\Phi_{p_0,\Delta+1,S}$, Q.E.D.

Notice that the estimate (4.15) is in general case non-improvable, see [21].

Remark 4.3. The right-hand side inequality (4.7) of theorem 4.1 is non-improvable. Namely, for the random variable η , defined on the sufficiently rich probabilistic space with the tail behavior

$$T_\eta(t) \sim t^{-p_0} \log^\Delta(t) S(\log t), \quad t \rightarrow \infty \quad (4.16)$$

we have as $p \rightarrow p_0 - 0 \Rightarrow \|\eta\|_p \asymp$

$$\Gamma(1 + \Delta) (p_0 - p)^{-(1+\Delta)/p_0} S^{1/p_0}(1/(p_0 - p)) = \Gamma(1 + \Delta) \psi_{p_0,\Delta,S}(p). \quad (4.17)$$

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